Sum formula for double Eisenstein series

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Theorem : Sum formula for double zeta values

For $\zeta(r,s) = \sum_{m>n>0} \frac{1}{m^r n^s}$, with $r \geq 2$ and $s \geq 1$, we have

$$\sum_{i=1}^{k-2} \zeta(k-i,i) = \zeta(k), \quad \sum_{i=1}^{k-2} 2^{k-i}\zeta(k-i,i) = (k+1)\zeta(k).$$

Theorem : Sum formula for double Eisenstein series

For double Eisenstein series $G_{r,s}(\tau) = \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbb{Z}\tau+\mathbb{Z} \\ \mathbf{m}\succ\mathbf{n}\succ 0}} \frac{1}{\mathbf{m}^r\mathbf{n}^s}$, τ in the upper half-plane, we have

$$\sum_{i=1}^{k-2} G_{k-i,i}(\tau) = G_k(\tau), \quad \sum_{i=1}^{k-2} 2^{k-i}G_{k-i,i}(\tau) = (k+1)G_k(\tau).$$

Sum formula for double Eisenstein series of level 2

Theorem (Kaneko-Tsumura) Sum formula for double T-values

For $T(r,s) = \sum_{\substack{m>n>0 \\ m:\text{ even},\, n:\text{ odd}}} \frac{4}{m^r n^s}$, $r \geq 2$ and $s \geq 1$, we have

$$\sum_{i=1}^{k-2} 2^{k-i-1} T(k-i,i) = (k-1)T(k).$$

Remark

Use notation $\zeta^{\mathbf{eo}}(r,s) = \sum_{\substack{m>n>0 \\ m:\text{ even},\, n:\text{ odd}}} \frac{1}{m^r n^s}$ and $\zeta^{\mathbf{o}}(k) = \sum_{n>0,\text{ odd}} \frac{1}{n^k}$ by Kaneko and Tasaka, we have

$$\sum_{i=1}^{k-2} 2^{k-i} \zeta^{\mathbf{eo}}(k-i,i) = (k-1)\zeta^{\mathbf{o}}(k).$$

Sum formula for double Eisenstein series of level 2

Definition (Kaneko-Tasaka)

$$G^{\mathbf{eo}}_{r,s}(\tau) := (2\pi i)^{-r-s} \sum_{\substack{\lambda>\mu>0 \\ \lambda\in\mathbf{ev}\cdot\tau+\mathbf{ev} \\ \mu\in\mathbf{ev}\tau+\mathbf{od}}} \frac{1}{\lambda^r\mu^s} \quad (r\geq 3, s\geq 2).$$

$$G^{\mathbf{o}}_{k}(\tau) := (2\pi i)^{-k} \sum_{\substack{\lambda>0 \\ \lambda\in\mathbf{ev}\cdot\tau+\mathbf{od}}} \frac{1}{\lambda^k}.$$

GOAL

$$\sum_{i=1}^{k-2} 2^{k-i} G^{\mathbf{eo}}_{k-i,i}(\tau) = (k-1)G^{\mathbf{o}}_{k}(\tau).$$

Theorem (Kaneko-Tasaka) q-expansion of level 2 double Eisenstein series

$$G^{\mathbf{eo}}_{r,s}(\tau) = \tilde{\zeta}^{\mathbf{eo}}(r,s) + g^{\mathbf{eo}}_{r,s}(q) + \text{modified term for } (r=1,2, s=1)$$

$$+ \sum_{\substack{p+h=k \\ p>1}} \left\{ \left((-1)^s \binom{p-1}{s-1} + \delta_{p,s} \right) \tilde{\zeta}^{\mathbf{o}}(p) g^{\mathrm{e}}_h(q) + (-1)^{p+r} \binom{p-1}{r-1} \tilde{\zeta}^{\mathbf{o}}(p) g^{\mathbf{o}}_h(q) \right\},$$

$$G^{\mathrm{o}}_k(q) = \tilde{\zeta}^{\mathrm{o}}(k) + g^{\mathrm{o}}_k(q).$$

where $g^{\mathbf{e}}_r(q) = \dfrac{(-1)^r}{2^r(r-1)!} \sum_{u,m>0} u^{r-1} q^{um}, g^{\mathbf{o}}_r(q) = \dfrac{(-1)^r}{2^r(r-1)!} \sum_{u,m>0} (-1)^u u^{r-1} q^{um}$,

and $\tilde{\zeta}^{**}(r,s) = (2\pi i)^{-r-s} \zeta^{**}(r,s)$, $\tilde{\zeta}^{*}(k) = (2\pi i)^{-k} \zeta^{*}(k)$.

Outline of the proof

1. Use some binomial coefficient identities to get

$$\begin{aligned} LHS &= \sum_{i=1}^{k-2} 2^{k-i} \zeta^{\mathbf{eo}}(k-i,i) + 4 \sum_{\substack{i=1 \\ i:\ \text{even}}}^{k-2} \tilde{\zeta}^{\mathbf{o}}(i) g^{\mathbf{o}}_{k-i} + \sum_{i=1}^{k-2} 2^{k-i} g^{\mathbf{eo}}_{i,k-i} \\ RHS &= (k-1)(\tilde{\zeta}^{\mathbf{o}}(k) + g^{\mathbf{o}}_k(q)) + \text{modified term} \end{aligned}$$

2. Express $\tilde{\zeta}^{\mathbf{o}}(j)$ as $(1-\frac{1}{2^j})\cdot(-\frac{1}{2})\cdot(\frac{B_j}{j!})$, and then apply the properties of Bernoulli numbers and Bernoulli polynomial.
3. For a fixed n, we compare the coefficients of q^n. One key is a combinatorial identity (by Liouville) and its generalization and variants.

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (f(a-b) - f(a+b))$$

$$= f(0)(\sigma(n) - d(n)) + \sum_{\substack{d\in\mathbb{N} \\ d|n}} \left(1 + \frac{2n}{d} - d\right) f(d) - 2\sum_{\substack{d\in\mathbb{N} \\ d|n}} \left(\sum_{v=1}^{d} f(v)\right).$$

Thank you for listening.
ご清聴ありがとうございました。