Formal finite multiple zeta values and modular forms

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$$\mathcal{Z}^{\mathcal{A}} \xrightarrow{\overset{?}{\cong}} \mathcal{Z} \Big/ \zeta(2)\mathcal{Z}$$

$$\mathrm{Fil}_4^{\mathrm{dep}} \mathcal{F}_k \xrightarrow{\overset{?}{\cong}} \mathcal{D}_k \Big/ \mathcal{P}_k$$

2nd Kindai Workshop (Multiple zeta values and modular forms), November 8th, 2024

Slides are available at **risan.io**

About me

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Classical MZV & finite MZV

MZV

$$\zeta(k_1,\ldots,k_r)=\sum_{m_1>\cdots>m_r>0}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\in\mathbb{R}\,.$$

- $\mathcal{Z}$: the $\mathbb{Q}$-vector space of MZVs,
- $\mathcal{Z}_k\subset\mathcal{Z}$: $\mathbb{Q}$-subspace of weight k MZVs.

Finite MZV

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r)=\left(\sum_{p>m_1>\cdots>m_r>0}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\mod p\right)_{p\text{ prime}}\in\mathcal{A}=\prod_{p\text{ prime}}\mathbb{F}_p\Big/\bigoplus_{p\text{ prime}}\mathbb{F}_p\,.$$

- $\mathcal{Z}^{\mathcal{A}}$: $\mathbb{Q}$-vector space of finite MZVs,
- $\mathcal{Z}_k^{\mathcal{A}}\subset\mathcal{Z}^{\mathcal{A}}$: subspace of weight k finite MZVs.

Kaneko-Zagier conjecture & relation to cusp forms

Conjecture (Kaneko-Zagier)

There exists an isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{Z}^{\mathcal{A}} \longrightarrow \mathcal{Z}/\pi^2\mathcal{Z}$$

$$\text{finite MZV of depth } r \longmapsto \text{MZV depth } r-1$$

Proposition/Conjecture (Gangl-Kaneko-Zagier, 2006)

Cusp forms of weight $2(a+b+1)$ give (conjecturally all) relations among $\zeta(2a+1, 2b+1)$.

Example:

$$\Delta \quad \rightsquigarrow \quad 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3) \equiv 0 \pmod{\pi^2\mathcal{Z}}$$

Question

What are the corresponding relations among depth 3 finite MZVs?

Modular forms and finite MZV

Observation (Kaneko-Zagier)

The relations among $\zeta_{\mathcal{A}}(2a,1,2b,1)$ seem to correspond to cusp forms of weight $2(a+b+1)$.

Example:

$$\Delta \overset{?}{\rightsquigarrow} 16\zeta_{\mathcal{A}}(2,1,8,1)+9\zeta_{\mathcal{A}}(4,1,6,1)+18\zeta_{\mathcal{A}}(6,1,4,1)-2\zeta_{\mathcal{A}}(8,1,2,1)=0\,.$$

This might lead to an answers of the previous question because of the following theorem.

Theorem (R, 2024+)

For even $k \geq 4$,

$$\mathrm{Fil}_4^{\mathrm{dep}}\,\mathcal{Z}_k^{\mathcal{A}} = \mathrm{Fil}_3^{\mathrm{dep}}\,\mathcal{Z}_k^{\mathcal{A}}.$$

$\mathrm{Fil}_d^{\mathrm{dep}}\,\mathcal{Z}_k^{\mathcal{A}}$: subspace finite MZVs of weight k and depth $\leq d$.

Conjecture (Formal analogue of Kaneko-Zagier conjecture in depth 2 - R., 2024+)

For even $k \geq 4$ there exists an isomorphism of $\mathbb{Q}$-algebras

$$\mathrm{Fil}_4^{\mathrm{dep}}\, \mathcal{F}_k \longrightarrow \mathcal{D}_k \big/ \mathcal{P}_k.$$

- $\mathrm{Fil}_d^{\mathrm{dep}}\, \mathcal{F}_k$: the space of **formal finite MZVs** of weight k and depth $\leq d$.
  (Spanned by symbols $\zeta_{\mathcal{A}}^{f}(k_1, \dots, k_r)$ satisfying conjecturally the same relations as finite MZV)
- $\mathcal{D}_k$: the **formal double zeta space** as introduced by Gangl-Kaneko-Zagier.
  (Spanned by symbols Z_k, $Z_{r,s}$, $P_{r,s}$ satisying the double shuffle relations)
- $\mathcal{P}_k$: the formal analog of $\pi^2\mathcal{Z}$ in depth 2 defined by

$$\mathcal{P}_k = \underbrace{\mathbb{Q}Z_k}_{\text{"}\zeta(k)\text{"}} + \underbrace{\mathbb{Q}Z_{1,k-1}}_{\text{"nonadmissible"}} + \underbrace{\langle P_{r,k-r} : r \text{ even}\rangle_{\mathbb{Q}}}_{\text{"}\zeta(2a)\zeta(2b)\text{"}}.$$

Modular forms and finite MZV

Proposition (R., 2024+)

For even $k \geq 4$,

$$\dim_{\mathbb{Q}} \mathcal{D}_k \Big/ \mathcal{P}_k = \frac{k}{2} - 2 - \dim_{\mathbb{C}} S_k.$$

Conjecture (R.)

For even $k \geq 4$,

$$\dim_{\mathbb{Q}} \mathrm{Fil}_4^{\mathrm{dep}} \mathcal{F}_k = \frac{k}{2} - 2 - \dim_{\mathbb{C}} S_k.$$

Moreover, $$\mathrm{Fil}_4^{\mathrm{dep}} \mathcal{F}_k = \sum_{\substack{a,b \geq 1 \\ 2(a+b+1)=k}} \mathbb{Q}\zeta_{\mathcal{A}}^{f}(2a,1,2b,1) = \sum_{\substack{a,b \geq 1 \\ 2(a+b+1)=k}} \mathbb{Q}\zeta_{\mathcal{A}}^{f}(1,2a,2b+1).$$

Conjecture $\Rightarrow$ There are $\dim_{\mathbb{Q}} S_{2(a+b+1)}$ relations among $\zeta_{\mathcal{A}}^{f}(2a,1,2b,1)$ (resp. $\zeta_{\mathcal{A}}^{f}(1,2a,2b+1)$).