$\sum$ and $\int$ – discretization, q-analogues, and related topics –

Yuto Tsuruta (Tohoku Univ.)

2nd Kindai Workshop Multiple Zeta Values and Modular Forms

My target – multiple zeta value

Multiple zeta value $\zeta(\boldsymbol{k}) := \zeta(k_1, \ldots, k_r)$ has two types of representations:

$$\sum_{0<m_1<\cdots<m_r} \frac{1}{\underbrace{m_1 \cdots m_1}_{k_1} \cdots \underbrace{m_r \cdots m_r}_{k_r}}.$$

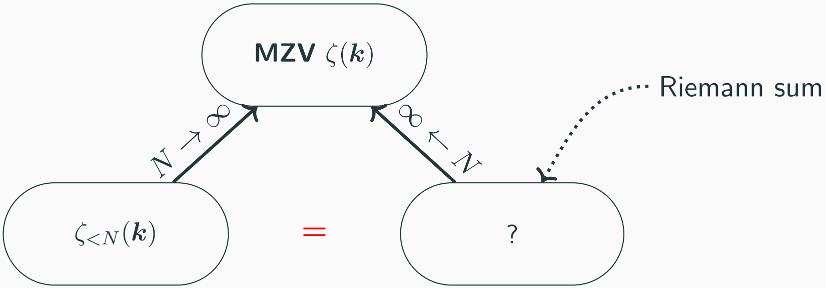
My target – multiple zeta value

Multiple zeta value $\zeta(\boldsymbol{k}) := \zeta(k_1, \ldots, k_r)$ has two types of representations:

$$\int\cdots\int_{\substack{0<t_{j1}<\cdots<t_{jk_j}<1\ (1\le j\le r)\\ t_{jk_j}<t_{(j+1)1}\ (1\le j<r)}} \underbrace{\omega_1(t_{11})\omega_0(t_{12})\cdots\omega_0(t_{1k_1})}_{k_1}\cdots\underbrace{\omega_1(t_{r1})\omega_0(t_{r2})\cdots\omega_0(t_{rk_r})}_{k_r}.$$

Here, $\omega_0(t) = \dfrac{dt}{t}$, $\omega_1(t) = \dfrac{dt}{1-t}$.

DISCRETIZATION



Question

Can we find suitable Riemann sum satisfies $=$?

DISCRETIZATION

Theorem (MSW formula, Maesaka–Seki–Watanabe)

For any index $\boldsymbol{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{>0})^r$ and any $N > 0$, it holds

$$\left(\sum\text{-side}\right)\ \zeta_{<N}(\boldsymbol{k}) = \zeta^{\flat}_{<N}(\boldsymbol{k})\ \left(\int\text{-side}\right).$$

DISCRETIZATION

Theorem (MSW formula, Maesaka–Seki–Watanabe)

For any index $\boldsymbol{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{>0})^r$ and any $N > 0$, it holds

$$\left(\sum\text{–side}\right)\ \zeta_{<N}(\boldsymbol{k}) = \zeta^{\flat}_{<N}(\boldsymbol{k})\ \left(\int\text{–side}\right).$$

$$\zeta^{\flat}_{<N}(\boldsymbol{k}) = \sum_{\substack{0<n_{j1}\leq\cdots\leq n_{jk_j}<N\ (1\leq j\leq r)\\ n_{jk_j}<n_{(j+1)1}\ (1\leq j<r)}} \prod_{i=1}^{r} \frac{1}{(N-n_{i1})n_{i2}\cdots n_{ik_i}}.$$

My result

Key Theorem (q-analogue of MSW formula, T., 24+)

For any diagonally constant index $\boldsymbol{k}$ and any $N > 0$, we have

$$\left(\sum \text{-side}\right)\ \zeta^{BZ}_{<N}(\boldsymbol{k}) = \zeta^{q\flat}_{<N}(\boldsymbol{k})\ \left(\int \text{-side ?}\right).$$

This theorem gives a generalization of MSW formula and Yamamoto's result.

Application and my interests

Application

Key Theorem provides a new proof of a relation like **Hoffman duality**.

Application and my interests

Application

Key Theorem provides a new proof of a relation like **Hoffman duality**.

My interests

Does Key Theorem play an important role in the study of q-MZV?

Application and my interests

Application

Key Theorem provides a new proof of a relation like **Hoffman duality**.

My interests

Does Key Theorem play an important role in the study of q-MZV?

Introducing myself

I recently have an interest in the irrationality and transcendence of numbers!