Central corrections on dimorphic groups

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Formal MZVs

The formal multiple zeta values (via Racinet, Bachmann–van-Ittersum–Matthes, etc.):

$$\mathcal{Z}^{\mathfrak{f}} := (\mathfrak{H}^1, *)/(\text{regularized double shuffle}),$$

with $(\mathfrak{H}^1, *)$: *harmonic algebra* (Hoffman).

Main Conjecture

$$\mathrm{span}_{\mathbb{Q}}\{\text{multiple zeta values}\} \overset{?}{\simeq} \mathcal{Z}^{\mathfrak{f}} \qquad (\text{as } \mathbb{Q}\text{-algebras})$$

Ecalle's terminology – symmetries on moulds

Mould: "function of a variable number of variables". The space of all moulds admit an addition $+$ and multiplication $\times$.

Example (Generating functions of MZVs)

$$\mathrm{Zag}(u_1,\dots,u_r) \overset{\text{reg.}}{:=} \int_{0<t_1<\cdots<t_r<1} \frac{t_1^{-u_1}}{1-t_1}dt_1 \cdots \frac{t_r^{-u_r}}{1-t_r}dt_r,$$

$$\mathrm{Zig}(v_1,\dots,v_r) \overset{\text{reg.}}{:=} \sum_{0<n_1<\cdots<n_r} \frac{1}{(n_1-v_r)\cdots(n_r-v_1)}$$

Famous algebraic relations among MZVs of type "$\zeta(\boldsymbol{k})\zeta(\boldsymbol{l}) = \sum_{\boldsymbol{h}} \zeta(\boldsymbol{h})$" are translated in mould theory, as *mould symmetries*.

Usual words	in Ecalle's theory
Shuffle relation (for itr. int.)	symmetral
Harmonic (or stuffle) relation (for nested sum)	symmetril

Dimorphic group

Generating functions Zag (of $\int$-type) and Zig (of $\sum$-type) are related by swap:

$$\mathrm{Zig} = \mathrm{Mini} \times \mathrm{swap}(\mathrm{Zag}),$$

where Mini is a constant mould (i.e., independent of all variables).
Such an operation (like multiplying Mini) is called a central correction.

Definition (Dimorphic group of as $*$ is type)

We define *monomorphic* (= having a simple mould symmetry) groups as

$$\mathrm{GARI}_{\mathrm{as}} := \{M \mid \text{symmetral}\}, \qquad \mathrm{GARI}_{\mathrm{is}} := \{M \mid \text{symmetril}\}$$

and, "dimorphic" (= having symmetries with/without swapping) group admitting central correction as

$$\mathrm{GARI}_{\mathrm{as}*\mathrm{is}} := \{M \in \mathrm{GARI}_{\mathrm{as}} \mid C \times \mathrm{swap}(M) \in \mathrm{GARI}_{\mathrm{is}} \ (\exists C\text{:const. mould})\}.$$

Main result

Theorem

A central correction in $\mathrm{GARI}_{\mathrm{as*is}}$ *is unique.*
Namely, for any symmetral M*, a constant mould* C *which makes* $C \times \mathrm{swap}(M)$ *symmetril is unique (if it exists).*

This theorem shows that, $\mathrm{GARI}_{\mathrm{as*is}}$ and $\mathcal{Z}^{\mathfrak{f}}$ are essentially equivalent notions.

Bonus: Expected future works

1. Does a similar fact exist in the Lie algebra $\mathrm{ARI}_{\mathrm{al}*\mathrm{il}}$ of $\mathrm{GARI}_{\mathrm{as}*\mathrm{is}}$?
2. Ecalle showed that, using the so-called flexion unit $\mathfrak{E}$, we can construct a new mould $\mathfrak{ess}_{\mathfrak{E}}$, and there exist an isomorphism

$$\mathrm{GARI}_{\mathrm{as}*\mathrm{as}} \xrightarrow{\mathrm{adgari}(\mathfrak{ess})} \mathrm{GARI}_{\mathrm{as}*\mathfrak{os}}.$$

   (Today's case: "polar specialization" $\mathfrak{E} = \mathrm{Pi} = \mathrm{swap}(\mathrm{Pa})$)
   Then, how does our result become in the space $\mathrm{GARI}_{\mathrm{as}*\mathrm{as}}$ of "bisymmetral" mould?
3. Can we construct "mould theoretical definition" of q-harmonic/shuffle relations of q-MZVs? (Ecalle's *twisted symmetries* by flexion units may work?)