Coaction Formula for the motivic version of Yamamoto's integral

Ku-Yu Fan

2024/11/9

1. Iterated integral
2. Yamamoto's integral
3. The motivic version
4. Coaction fomula
5. Examples
6. References

Iterated integral

Multiple zeta values(MZVs)

Definition

Multiple zeta values is a real number defined by

$$\zeta(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}},$$

where $k_1, \ldots, k_{d-1} \in \mathbb{Z}_{>0}, k_d \in \mathbb{Z}_{>1}$.

Iterated integral

Definition

For $a_0, \ldots, a_{k+1} \in \mathbb{C}$ with $a_0 \neq a_1, a_k \neq a_{k+1}$, and a piecewisely smooth path $\gamma : [0,1] \to \mathbb{C}$ from a_0 to a_{k+1} such that $\gamma((0,1)) \subset \mathbb{C} \setminus \{a_1, \ldots, a_k\}$, we define

$$I_\gamma(a_0; a_1, \ldots, a_k; a_{k+1})$$

by the **iterated integral**

$$\int_{0<t_1<\cdots<t_k<1} \prod_{j=1}^{k} \frac{d\gamma(t_j)}{\gamma(t_j) - a_j} \in \mathbb{C}.$$

Remark (Iterated integral expression)

$$\zeta(k_1, \ldots, k_d) = (-1)^d I(0; 1, \{0\}^{k_1-1}, \ldots, 1, \{0\}^{k_d-1}; 1).$$

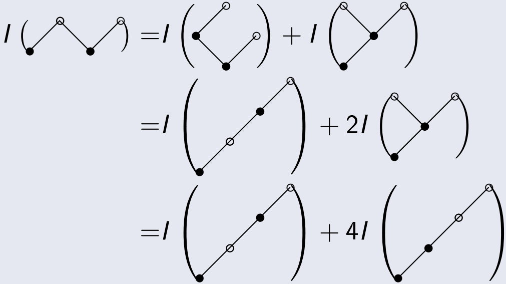
Yamamoto's integral

Yamamoto's integral

Example

I  $= (-1)I(0;1,0;1) = \zeta(2)$

Example



Yamamoto's integral

Example

For $p, q, a, b, c, d \in \mathbb{C}$

$$I_\gamma(X) = I_\gamma \left(\begin{array}{ccccc} & & & & d \\ & & & / & \\ & & c & & \\ & / & & \backslash & \\ a & & & & b \end{array} \right)$$

$$= I_\gamma(p; a, b, c, d; q) + I_\gamma(p; b, a, c, d; q).$$

The motivic version

Motivic iterated integrals

Let $\mathcal{P}$ be the $\mathbb{Q}$-algebra of the ring of periods of mixed Tate motives over $\mathbb{Q}$. Brown defined the motivic iterated integrals

$$I^{\mathfrak{m}}_{\gamma}(a_0; a_1, \ldots, a_k; a_{k+1}) \in \mathcal{P}.$$

It is known that there is a ring homomorphism called period map, which is onto and speculated to be injective.

$$\begin{array}{rccc} \text{per}: & \mathcal{P} & \to & \mathbb{C} \\ & I^{\mathfrak{m}}_{\gamma}(a_0; \ldots; a_{k+1}) & \mapsto & I_{\gamma}(a_0; \ldots; a_{k+1}) \end{array}$$

Motivic version of Yamamoto's integral

Definition

Motivic version of Yamamoto's integral $I_\gamma^{\mathfrak{m}}(X)$ is defined by

$$\begin{array}{ccc} I_\gamma^{\mathfrak{m}}(X) & := & \sum_{Y \in \mathrm{Tot}(X)} I_\gamma^{\mathfrak{m}}(Y) \\ \downarrow & & \downarrow \\ I_\gamma(X) & = & \sum_{Y \in \mathrm{Tot}(X)} I_\gamma(Y) \end{array}$$

where $\mathrm{Tot}(X)$ is the set of all totally order which we need to consider.

Coaction fomula

Coaction on $\mathcal{P}$

Let $\pi : \mathcal{P} \to \mathcal{A} := \mathcal{P}/(2\pi i)\mathcal{P}$ be the natural projection to its quotient. By the theory of mixed Tate motives, there is a natural motivic coaction $\Delta : \mathcal{P} \to \mathcal{A} \otimes \mathcal{P}$. Goncharov proved the coproduct formula of motivic iterated integrals, and Brown proved its coaction version.

Theorem (Goncharov)

$$\Delta(I^{\mathfrak{m}}_{\gamma}(a_0; a_1, \ldots, a_k; a_{k+1})) =$$

$$\sum_{s=0}^{k} \sum_{\substack{i_0 < \cdots < i_{s+1} \\ i_0 = 0 \\ i_{s+1} = k+1}} \prod_{p=0}^{s} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \otimes I^{\mathfrak{m}}_{\gamma}(a_{i_0}; a_{i_1}, \ldots, a_{i_s}; a_{i_{s+1}}).$$

Infinitesimal coaction

It is known that $\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}_k$ has a structure of graded algebra. Let $\pi' : \mathcal{A} \to \mathfrak{L} := \mathcal{A}/\mathcal{A}^2_{>0}$. be the natural projection to its quotient, where $\mathcal{A}_{>0}$ is $\bigoplus_{k>0} \mathcal{A}_k$. Brown also proved the infinitesimal coaction D_r which is given by the following formula.

Definition

For $r \in \mathbb{N}$, the infinitesimal coaction $D_r : \mathcal{P} \to \mathfrak{L} \otimes \mathcal{P}$ is defined by

$$D_r(I^{\mathfrak{m}}_{\gamma}(a_0; a_1, \dots, a_k; a_{k+1})) = \sum_{s=0}^{k-r} I^{\mathfrak{l}}(a_s; a_{s+1}, \dots, a_{s+r}; a_{s+r+1}) \otimes I^{\mathfrak{m}}_{\gamma}(a_0; a_1, \dots, a_s, a_{s+r+1}, \dots, a_k; a_{k+1})$$

Main theorem

Main Theorem 1 (F.)

$$D_r(I^{\mathfrak{m}}_\gamma(X)) = \sum_{\substack{Y \in X_r \\ Y \text{ is irr.}}} \sum_{\substack{p \to_{\tilde{X}} Y \\ q \leftarrow_{\tilde{X}} Y}} I^{\mathfrak{l}}_{(p;q)}(Y) \otimes I^{\mathfrak{m}}_\gamma(X_{(p,\widehat{Y},q)}).$$

Main Theorem 2 (F.)

This formula requires some conditions related to Y.

$$\Delta_Y(I^{\mathfrak{m}}_\gamma(X)) = \prod_{Z \in C_X(Y)} I^{\mathfrak{a}}_{(p_Z;q_Z)}(Z) \otimes I^{\mathfrak{m}}_\gamma(X_{\widehat{Y}}).$$

Remark

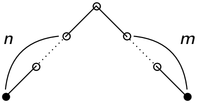
$$\Delta(I^{\mathfrak{m}}_\gamma(X)) = \sum_{Y \subset X} \Delta_Y(I^{\mathfrak{m}}_\gamma(X)).$$

Examples

Examples of Theorem 2

Let us calculate

$$\Delta \left(I^{\mathfrak{m}} \left(\right) \right).$$



using Theorem 2. By definition of motivic version of Yamamoto's integral, we have

$$\Delta(I^{\mathfrak{m}}(X)) = \sum_{i=1}^{n} \binom{n-i+m}{n-i} \Delta(I^{\mathfrak{m}}(0;1,\{0\}^{i},1,\{0\}^{(n-i)+m+1};1))$$
$$+ \sum_{j=1}^{m} \binom{n+m-j}{m-j} \Delta(I^{\mathfrak{m}}(0;1,\{0\}^{i},1,\{0\}^{n+(m-j)+1};1)).$$

Examples of Theorem 2

First, using the basic properties of motivic iterated integral, it is easy to show the formula

$$I^{\mathfrak{m}}(0; \{0\}^a, 1, \{0\}^b; 1) = (-1)^{a+1} \binom{a+b}{a} \zeta^{\mathfrak{m}}(a+b+1),$$

and using $\binom{i}{k} = \binom{i-1}{k} + \binom{i-1}{k-1}$, it is easy to show the formula

$$\sum_{k=0}^{i} (-1)^{i-k+1} \frac{m}{m+k} \binom{i}{k} = \frac{(-1)^{i+1}}{\binom{i+m}{m}}.$$

Motivic iterated integrals have the following property

$$I^{\mathfrak{m}}(0; 0; 1) = I^{\mathfrak{m}}(0; 1; 1) = I^{\mathfrak{m}}(a; a_1, \ldots, a_k; a) = 0. \ (k \geq 1).$$

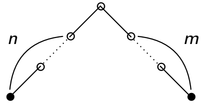
Next, using the remark of coproduct to have

$$\Delta(I^{\mathfrak{m}}(X)) = \sum_{Y \subset X} \Delta_Y(I^{\mathfrak{m}}(X)).$$

Examples of Theorem 2

Finally, using Theorem 2 and the formula in the previous page. We have

$$\Delta\left(I^{\mathfrak{m}}\left(\right.\right.$$



$$\left.\left.\right)\right)$$

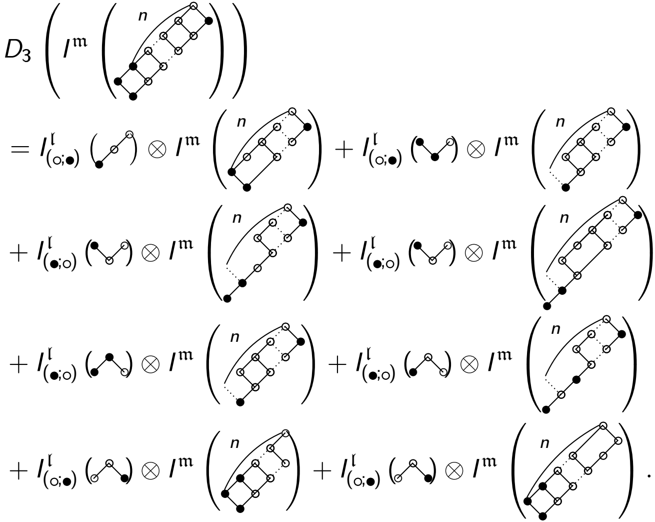
$$= \sum_{Y \subset X} \Delta_Y(I^{\mathfrak{m}}(X))$$

$$= 1 \otimes I^{\mathfrak{m}}(X) + I^{\mathfrak{a}}(X) \otimes 1$$

$$+ \sum_{i=0}^{n-2} (-1)^i I^{\mathfrak{a}}(0; 1, \{0\}^{m+i+1}; 1) \otimes I^{\mathfrak{m}}(0; 1, \{0\}^{n-i}; 1)$$

$$+ \sum_{j=0}^{m-2} (-1)^j I^{\mathfrak{a}}(0; 1, \{0\}^{n+j+1}; 1) \otimes I^{\mathfrak{m}}(0; 1, \{0\}^{m-j}; 1).$$

Examples of Theorem 1



References

References

- Ku-Yu Fan. Coproduct Formula for Motivic Version of Yamamoto's Integral. (2024). https://doi.org/10.48550/arXiv.2406.08127. https://doi.org/10.6342/NTU202401087.

Thank you for your attention!