On evaluations of multiple zeta-star values of Bowman-Bradley type

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Notation

$$\zeta^\star(k_1,\dots,k_r) := \sum_{0<n_1\le\cdots\le n_r} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}.$$

$ш$:index shuffle

$$\begin{aligned}\zeta^\star((a,b)\,ш\,(c)) &= \zeta^\star((a,b,c)+(a,c,b)+(c,a,b))\\ &= \zeta^\star(a,b,c)+\zeta^\star(a,c,b)+\zeta^\star(c,a,b).\end{aligned}$$

$(k_1,\dots,k_r)_+ := (k_1,\dots,k_{r-1},k_r+1)$.
$(k_1,\dots,k_r)^\vee$:Hoffman dual
For example,

$$((1,2)+(3))^\vee = (1,2)^\vee + (3)^\vee = (2,1)+(1,1,1).$$

Main conjecture

Conjecture (Maesaka)

Let $0 \leq n, m \in \mathbb{Z}$, then

$$\zeta^\star((((3, \{5,3\}^n) ш \{2\}^m)^\vee)_+) \overset{?}{\in} \mathbb{Q}\pi^{8n+2m+4}$$

and

$$\zeta^\star((((3, \{5,3\}^n) ш \{4\}^m)^\vee)_+) \overset{?}{\in} \mathbb{Q}\pi^{8n+4m+4}$$

hold. Especially ($m = 0$),

$$\zeta^\star(1,1,2,\{1,1,1,2,1,2\}^n) = \zeta^\star(((3,\{5,3\}^n)^\vee)_+) \overset{?}{\in} \mathbb{Q}\pi^{8n+4}.$$

Example

For example, the cases $m = 0, n = 1, 2, 3$ are

$$\zeta^\star(1,1,2,1,1,1,2,1,2) = \frac{529}{154791000}\pi^{12} = \frac{23^2}{2^3 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 13}\pi^{12},$$

$$\zeta^\star(1,1,2,\{1,1,1,2,1,2\}^2) \stackrel{?}{=} \frac{4423}{12281339070000}\pi^{20} = \frac{4423}{2^4 \cdot 3^8 \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17}\pi^{20},$$

$$\zeta^\star(1,1,2,\{1,1,1,2,1,2\}^3) \stackrel{?}{=} \frac{6400899142181}{16864322650412944707000000}\pi^{28}$$

$$= \frac{37 \cdot 239 \cdot 7541 \cdot 95987}{2^7 \cdot 3^{13} \cdot 5^7 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29}\pi^{28}.$$

Example

Examples of the cases $m > 0$:

$$\zeta^\star((((3,5,3)ш(2))^\vee)_+) = \frac{\pi^{14}}{801900} = \frac{\pi^{14}}{2^2\cdot 3^6\cdot 5^2\cdot 11},$$

$$\zeta^\star((((3,5,3)ш(4))^\vee)_+) = \frac{38593}{260513253000}\pi^{16} = \frac{38593}{2^3\cdot 3^7\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17}\pi^{16},$$

$$\zeta^\star((((3,5,3)ш(4,4))^\vee)_+) \stackrel{?}{=} \frac{51793}{13158577575000}\pi^{20} = \frac{7^3\cdot 151}{2^3\cdot 3^9\cdot 5^5\cdot 11^2\cdot 13\cdot 17}\pi^{20}.$$

Main theorem

The main result is as follows.

Theorem (Maesaka)

Let $0 \le n \in \mathbb{Z}$, then we have

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \zeta^\star((((3,\{5,3\}^k) ш \{4\}^{n-2k})^\vee)_+) = 2^{n+1} \sum_{k=0}^{n+1} \zeta(\{1,3\}^k)\zeta^\star(\{1,3\}^{n+1-k}) \in \mathbb{Q}\pi^{4n+4}$$

and

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \zeta^\star(((2,(3,\{5,3\}^k) ш \{4\}^{n-2k})^\vee)_+) = 2^{n+1} \sum_{k=0}^{n} \zeta(3,\{1,3\}^k)\zeta^\star(3,\{1,3\}^{n-k})$$
$$\in \mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\dots].$$

Further conjecture

The following conjectures are analogues of the main conjecture.

Conjecture (Maesaka)

Let $n > 0$ and $m \geq 0$. Then we have

$$\zeta^\star(((\{3,5\}^n ш \{2\}^m)^\vee)_+) \overset{?}{\in} \sum_{k=2}^{4n+m} \mathbb{Q}\pi^{8n+2m-2k}\zeta(2k+1)$$

and

$$\zeta^\star(((\{3,5\}^n ш \{4\}^m)^\vee)_+) \overset{?}{\in} \sum_{k=1}^{2n+m} \mathbb{Q}\pi^{8n+4m-4k}\zeta(4k+1).$$

Explicit conjecture

Conjecture (Maesaka)

Define two sequences a_n and b_n by the following recurrence formula.

$$\begin{aligned} a_n &:= 6a_{n-1} - a_{n-2}, \qquad & a_0 := 1, a_1 := 1, \\ b_n &:= 6b_{n-1} - b_{n-2}, \qquad & b_0 := 0, b_1 := 1. \end{aligned}$$

Then,

$$\begin{aligned} \zeta^\star(1, \{1,2,1,1,1,2\}^n) &\stackrel{?}{=} \frac{(2 \cdot 4^n + (-1)^{n-1}(16b_n^2+1))\zeta(8n+1)}{4^{n-1}} \\ &\quad + 2\sum_{k=1}^{n} \frac{(-1)^k(4a_k^2-1)}{4^{k-1}} \zeta(8k-3)\zeta^\star(1,1,2,\{1,1,1,2,1,2\}^{n-k}) \end{aligned}$$

holds.

Regularized analogue

We denote the harmonic regularization for multiple zeta-star values as $\zeta^{*,\star}$.

Conjecture (Maesaka)

Let $0 \leq n \in \mathbb{Z}$, then

$$\zeta^{*,\star}(1,\{1,2,1,1,1,2\}^n,1) \overset{?}{\in} \mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\dots],$$
$$\zeta^{*,\star}(1,1,2,\{1,1,1,2,1,2\}^n,1) \overset{?}{\in} \mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\dots],$$
$$\zeta^{*,\star}(1,1,2,\{1,1,1,2,1,2\}^n,1,1,1) \overset{?}{\in} \mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\dots]$$

hold.

Example

For the case $n = 1$, we have

$$\begin{aligned}
\zeta^{*,\star}(1,1,2,1,1,1,2,1) &= \frac{\pi^{10}}{2835} - 18\zeta(5)^2, \\
\zeta^{*,\star}(1,1,2,1,1,1,2,1,2,1) &= \frac{99}{2}\zeta(13) - \frac{17}{30}\pi^4\zeta(9), \\
\zeta^{*,\star}(1,1,2,1,1,1,2,1,2,1,1,1) &= \frac{273}{4}\zeta(15) + \frac{33}{8}\pi^2\zeta(13) - \frac{3}{4}\pi^4\zeta(11) - \frac{1717}{7560}\pi^6\zeta(9) \\
&\quad + \frac{2}{945}\pi^{10}\zeta(5) + \frac{529}{464373000}\pi^{12}\zeta(3) - 36\zeta(5)^3.
\end{aligned}$$