Manin's iterated integrals and multiple Hecke L-functions

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- I generalized Choie–Ihara result to iterated integrals of modular forms.

For $f_i \in M_{k_i}(\Gamma_0(N), \chi_i)(i = 1, \ldots, n)$, and $f_i(z) = \sum_{n=0}^{\infty} a_n^{(i)} q^n$, $s_1, \ldots, s_n \in \mathbb{C}$,

Definition 1

- $I_{i\infty}^{0}\begin{pmatrix} f_1, \ldots, f_n \\ s_1, \ldots, s_n \end{pmatrix} :=$

$$\int_{i\infty}^{0} f_1(z_1) z_1^{s_1} \frac{dz_1}{z_1} \int_{i\infty}^{z_1} f_2(z_2) z_2^{s_2} \frac{dz_2}{z_2} \cdots \int_{i\infty}^{z_{n-1}} f_n(z_n) z_n^{s_n} \frac{dz_n}{z_n}.$$

- $L\begin{pmatrix} f_1, \ldots, f_n \\ s_1, \ldots, s_n \end{pmatrix} :=$

$$\sum_{m_1, \ldots, m_n > 0} \frac{a_{m_1}^{(1)} \ldots a_{m_n}^{(n)}}{(m_1 + \cdots + m_n)^{s_1} (m_1 + \cdots + m_{n-1})^{s_2} \ldots (m_1)^{s_n}}.$$

Theorem 2 (F.Brown)

$I_{i\infty}^{0}\binom{f_1, \ldots, f_n}{s_1, \ldots, s_n}$ is meromorphic function over $\mathbb{C}^n$, and we denote

$$Z\binom{f_1, \ldots, f_n}{s_1, \ldots, s_n} := N^{\frac{s_1+\cdots+s_n}{2}} I_{i\infty}^{0}\binom{f_1, \ldots, f_n}{s_1, \ldots, s_n}.$$

Then $Z\binom{f_1, \ldots, f_n}{s_1, \ldots, s_n}$ satisfies functional equation.

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Theorem 3

$I^0_{i\infty}\begin{pmatrix} f_1, \ldots, f_n \\ s, \alpha_2, \ldots, \alpha_n \end{pmatrix}$ can be expressed as an explicit sum of $L\begin{pmatrix} f_1, \ldots, f_n \\ s, \alpha_2, \ldots, \alpha_n \end{pmatrix}$.

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- Relationship between multiple zeta values and modular forms via this integral.
- Linear relations for the special values $\boldsymbol{I}_{i\infty}^{\mathbf{0}}$.
- The geometric interpretation of $\boldsymbol{I}_{i\infty}^{\mathbf{0}}$.