P-ordering sequence of the multiple zeta Functions

2nd Kindai Workshop MZV and Modular forms

Francis A. Howard
CIPMA-UNESCO

9th November 2024

Bhargava Factorial

The *Bhargava factorial over* S, denoted $k!_S$, is defined as

$$k!_S = \prod_p v_k(S, p).$$

In particular, for $S = \mathbb{Z}$ we obtain

$$k!_{\mathbb{Z}} = \prod_{p \in \mathbb{P}} w_p(k!) = \prod_{p \in \mathbb{P}} p^{p(k!)} = k!.$$

q-analogue of Bhargava factorial

Let $S \in \left\{ \frac{q^k - 1}{q - 1} : k \in \mathbb{N} \right\}$ in the ring $\mathbb{C}[q, q^{-1}]$ where $\left[k\right]_q = \frac{q^k - 1}{q - 1}$ where $k!$ is the Bhargava factorial given as

$$
\begin{aligned}
k!_S &= (q-1)^{-k}(q^k - 1)(q^{k-1} - 1) \cdots (q-1) \\
&= [k]_q! = [k]_q[k-1]_q \cdots [1]_q \\
&= \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q-1)}{(q-1) \cdots (q-1)^k},
\end{aligned}
$$

relation to q-analogue

the following binomials easily follow:

$$[n]_S! := n!_S,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_S := \frac{[n]_S!}{[k]_S![n-k]_S!} := \frac{n!_S}{k!_S(n-k)!_S}.$$

Bharagava Binomials

Definition (Exponential)

The generalized exponential function associated to an infinite subset E of $\mathbb{Z}$ is defined by

$$\exp_E(x) = \sum_{k=0}^{+\infty} \frac{1}{k!_E} x^k,$$

for $k \geq 0$, $k!$ divides $k!_E$ and this function converges everywhere.

Bhargava exponential

Definition (Cont.)

The generalized Euler number associated with E is given by

$$e_E = \exp_E(1) = \sum_{k=0}^{+\infty} \frac{1}{k!_E}.$$

Bharagava Euler exp.

Definition (Bernoulli numbers)

The generalized Bernoulli numbers $\mathbb{B}_{E,k}$ associated to an infinite subset E of $\mathbb{Z}$ is defined by

$$\frac{x}{\exp_E(x) - 1} = \sum_{k=0}^{+\infty} \frac{\mathbb{B}_{E,k}}{k!_E} x^k.$$

In particular, $1 \le e_E \le e$ and Mingarelli showed that e_E is always irrational.

Bhargava factorial for set of prime

Definition (Factorial sequence of the set of prime numbers)

Let p be prime, the Bhargava factorial for a set of prime $\mathbb{P}$ can also be computed using the following:

$$k!_{\mathbb{P}} = \prod_{p \in \mathbb{P}} p^{w_{\mathbb{P},p}(k)} = \prod_{p \in \mathbb{P}} p^{\sum_{i \geq 0} \left\lfloor \frac{k-1}{(p-1)p^i} \right\rfloor}$$

Adam et al. defn of Bhargava

equivalently we have, $k!_{\mathbb{P}} = k!e^{Ck+o(k)}$ where
$C = \sum_{p\in\mathbb{P}} \dfrac{\ln p}{(p-1)^2}$.
From this we obtain the formula:

$$\exp_{\mathbb{P}}(x) = \sum_{k=0}^{+\infty} \frac{1}{k!_{\mathbb{P}}} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{24}x^3 + \frac{1}{48}x$$

Bernoulli associated to $\mathbb{P}$

specifically, $\frac{5}{2} < e_{\mathbb{P}} < e$.
From the definition 1, we have the Bernoulli numbers associated with $\mathbb{P}$ $(\mathbb{B}_{\mathbb{P},n})$:

$$\frac{x}{\exp_{\mathbb{P}}(x) - 1} = \sum_{k=0}^{+\infty} \frac{\mathbb{B}_{\mathbb{P},k}}{k!_{\mathbb{P}}} x^k.$$

Few Values of Bernoulli number to $\mathbb{P}$

by induction one can easily find, $\mathbb{B}_{\mathbb{P},0} = 1$ for every $k \geq 1$ and $\sum_{r=0}^{k} \begin{bmatrix} k+1 \\ r \end{bmatrix}_{\mathbb{P}} \mathbb{B}_{\mathbb{P},r} = 0$ Following this , the first values of the Bernoulli number associated to set of prime $\mathbb{P}$ is given as:

$$\mathbb{B}_{\mathbb{P},0} = 1, \quad \mathbb{B}_{\mathbb{P},1} = -\frac{1}{2}, \quad \mathbb{B}_{\mathbb{P},2} = \frac{5}{12}, \quad \mathbb{B}_{\mathbb{P},3} = -\frac{5}{2}$$

Generalized Bernoulli polynomials

The generalized Bernoulli polynomials associated to $E \in \mathbb{Z}$ is defined by the power series:

$$\frac{x\exp_E(yx)}{\exp_E(x) - 1} = \sum_{k=0}^{+\infty} \mathbb{B}_{E,k}(y)\frac{x_k}{k!_E}.$$

We easily deduce the *kth* Bernoulli polynomial associated to $E \in \mathbb{Z}$ as:

$$\mathbb{B}_{E,k}(Y) = \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_E \mathbb{B}_{E,r} Y^{k-r}$$

Theorem

Theorem (Adam et al.)

For every k,

$$\mathbb{B}_{\mathbb{P},k}(Y) - \mathbb{B}_{\mathbb{P},k}(0) = \sum_{r=0}^{k-1} \begin{bmatrix} k \\ r \end{bmatrix}_{\mathbb{P}} \mathbb{B}_{\mathbb{P},r} Y^{k-r} \in \mathbb{Z}[Y],$$

whenever, $\frac{(r+1)!_{\mathbb{P}}}{r!_{\mathbb{P}}}$ *divides* $\left[\begin{smallmatrix} k+1 \\ r \end{smallmatrix}\right]_{\mathbb{P}}$ *for* $0 \leq r \leq k$.

Few values of Bernoulli poly.

From the theorem above we can have the first few Bernoulli polynomials associated to $\mathbb{P}$:

1. $\mathbb{B}_{\mathbb{P},1}(Y) = Y - \frac{1}{2}$,
2. $\mathbb{B}_{\mathbb{P},2}(Y) = Y^2 - Y + \frac{5}{12}$,
3. $\mathbb{B}_{\mathbb{P},3}(Y) = Y^3 - 6Y^2 + 5Y - \frac{5}{2}$,
4. $\mathbb{B}_{\mathbb{P},4}(Y) = Y^4 - Y^3 + 5Y^2 - 5Y + \frac{103}{40}$,

One can go on to list several examples.

Zeta Functions

The Riemann zeta function has always been concept of great study. It is defined as the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots .$$

for complex s. The series is convergent whenever $\mathrm{Re}(s) > 1$. This is related to prime numbers as follows:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

q-analogue of the Riemann zeta function

Kaneko et al. studied the q-analog of the Riemann zeta function

$$\zeta_q(s) = \sum_{k=1}^{\infty} \frac{q^{k(s-1)}}{[k]_q^s},$$

They found that for $0 < q < 1$, as a function of $(s, t) \in \mathbb{C}^2$, $f_q(s, t)$ is continued meromorphically via the series expansion of $f_q(s, t)$

Theorem

$$\lim_{q \uparrow 1} \zeta_q(-m) = -\frac{B_{m+1}}{m+1}.$$

for each non-negative integer m,

Theorem[Kaneko et al.]

For any $s \in \mathbb{C}$, $s \neq 1$, we have

$$\lim_{q \uparrow 1} \zeta_q(s) = \zeta(s).$$

Bradley and Zhao

Definition (multiple q-zeta values)

Let m be a positive integer and let $s_1, s_2, \dots, s_m$ be real numbers with $s_1 > 1$ and $s_j \geq 1$ for $2 \leq j \leq m$. The multiple q-zeta function is the nested infinite series:

$$\zeta_q[s_1, \cdots, s_m] := \sum_{k_1 > \cdots k_m > 0} \prod_{j=1}^{m} \frac{q^{k_j(s_j-1)}}{[k_j]_q^{s_j}},$$

Cont'd

where the sum is over all positive integers k_j satisfying the indicated inequalities. As

$$\lim_{q\to 1} \zeta_q[s_1, \cdots, s_m] := \zeta_q(s_1, \cdots, s_m) \quad (2)$$
$$:= \sum_{k_1>\cdots k_m>0}^{\infty} \prod \frac{1}{k_j^{s_j}}$$

we obtain the usual multiple zeta value.

Cont'd

Definition (q-Shuffle product)

For integers $s, t > 1$ we have

$$\begin{aligned}\zeta_q(s)\zeta_q(t) &= \zeta_q(s,t) + \zeta_q(t,s) + \zeta_q(s+t)(3)\\ &\quad + (q-1)\zeta_q(s+t-1),\end{aligned}$$

when $\lim_{q\uparrow 1} \zeta_q(s)\zeta_q(t) = \zeta(s)\zeta(t)$.

Main Theorem

Let $\mathbf{Int}(E,\mathbb{Z}) = \{f \in \mathbb{Q}[X] | f(E) \subseteq \mathbb{Z}\}$ be the ring of integer-valued polynomials on a subset E of $\mathbb{Z}$.

1. For every integer $k > 0$,

$$\lim_{q \uparrow 1} \zeta_q(-k) = -\frac{\mathbb{B}_{E,k+1}}{k+1}.$$

Remark

where $\mathbb{B}_{E,k}$ is the generalized Bernoulli polynomial associated to $E \subseteq \mathbb{Z}$.

Theorem cont'd

If we put $\mathbb{P}$ to be the set of prime numbers, and we define the Bernoulli associated to $\mathbb{P}$ we obtain

$$\lim_{q\uparrow 1} \zeta_q(-k) = -\frac{\mathbb{B}_{\mathbb{P},k+1}}{k+1}.$$

Theorem Cont'd

1. Let m_1, m_2 be two non-negative integers and $k = m_1 + m_2 + 2$. Then $\zeta(s_1, s_2)$ has indeterminacy at $(-m_1, -m_2)$ such that

Theorem Cont'd

$$\zeta(-m_1, -m_2)_q := (1-q)^{2-m_1}$$

$$\times\left\{\frac{(-1)^k}{(m_1+1)(m_2+1)(\log q)^2}\right.$$

$$+\sum_{r=0}^{m_1} \frac{(-1)^{r+m_2+1}}{(m_2+1)\log q} \begin{bmatrix} m_1 \\ r \end{bmatrix}_E \frac{1}{q^{m_1+1-r}-1}$$

$$+\sum_{r=0}^{m_1} \frac{(-1)^{r+m_1+1}}{\log q}$$

$$\times \frac{m_1!_E \quad (m_2+1-r)!_E}{(k-r)!_E} \begin{bmatrix} m_1 \\ r \end{bmatrix}_E \frac{1}{q^{m_1+1-r}-1}$$

Theorem Cont'd

$$+\sum_{r=0}^{m_1}\sum_{r=0}^{m_2}(-1)^{r_1+r_2}\begin{bmatrix} m_1 \\ r_1 \end{bmatrix}_E \begin{bmatrix} m_2 \\ r_2 \end{bmatrix}_E$$

$$\left.\times\frac{1}{q^{m_2+1-r_2}-1}\frac{1}{q^{k-r_1-r_2}-1}\right\}$$

J. Zhao

For non-integers m_1 and m_2, the following results holds for $E = \mathbb{Z}$:

$$\lim_{q \uparrow 1} \zeta(-k, -r) := \zeta(s_1, s_2)$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_E := \frac{n!_E}{k!_E (n-k)!_E}.$$

is the generalized Bhargava factorial.

Questions?

Thank you for your attentioin